A Colin de Verdiere-Type Invariant and Odd-K_4- and Odd-K_3^2-Free Signed Graphs

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Conference honouring Robin's 50th birthday

Joint work with Marina Arav, Frank Hall, and Jason Li

The parameter μ.

Definition

For a graph $G = (V, E)$ with n vertices, $O(G)$ is the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

1. $a_{i,j} < 0$ if i and j are adjacent,
2. $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent
3. $a_{i,i} \in \mathbb{R}$ for $i \in V$.

Definition (Y. Colin de Verdière)

$\mu(G)$ is the largest possible nullity (corank) of any $A \in O(G)$ that has exactly one negative eigenvalue and that satisfies the Strong Arnold Hypothesis (SAH).

Theorem

1. *$\mu(G) \leq 3$ if and only if G is planar. (Colin de Verdière)*
2. *$\mu(G) \leq 4$ if and only if G has a flat embedding. (Conjectured by Robertson, Seymour, Thomas. Proved by Lovász and Schrijver.)*

No characterization known for graphs G with $\mu(G) \leq 5$.

Conjecture (Luke Postle)

For all graphs H, there exists a c_H such that all graphs G with no H-minor have $\mu(G) \leq c_H$.

Support for this conjecture:

1. μ behaves well under clique sums,
2. behaves well under adding vertices, and
3. is bounded for graphs embedded on a fixed surface. But the problem is how to deal with vortices.

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Conjecture

μ is bounded on an ideal I if and only if I is a proper ideal.

Support for this conjecture:

1. μ behaves well under clique sums,
2. behaves well under adding vertices, and
3. is bounded for graphs embedded on a fixed surface. But the problem is how to deal with vortices.

The parameter ν

Definition

For a graph $G = (V, E)$ with n vertices, $S(G)$ is the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

1. $a_{i,j} \neq 0$ if i and j are adjacent,
2. $a_{i,j} = 0$ if $i \neq j$ and i and j are non-adjacent,
3. $a_{i,i} \in \mathbb{R}$ for all $i \in V$.

Definition (Colin de Verdière)

$\nu(G)$ is the largest possible nullity of any positive semidefinite matrix $A = [a_{i,j}] \in S(G)$ that satisfies the SAH.

Theorem

$\nu(G)$ is large if and only if tree-width G is large.

Can we generalize these to one graph parameter?

Signed graphs

Definition

A signed graph is a pair (G, Σ) with $\Sigma \subseteq E(G)$. (We allow parallel edges, but no loops.) Edges in Σ are called odd edges and the other edges are called even edges.

Definition

A cycle C in (G, Σ) is called odd if C has an odd number of odd edges.

Example

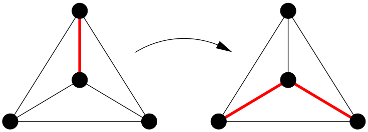
A signed graph (G, Σ) is called bipartite if it has no odd cycles.

Theorem (Zaslavsky)

(G, Σ) and (G, Ω) have the same set of odd cycle if and only if there is an $S \subseteq V(G)$ such that $\Omega = \Sigma \Delta \delta(S)$.

Definition

The operation $\Sigma \to \Sigma\Delta\delta(S)$ is called a signature-exchange.



Definition

For a signed graph (G, Σ) with n vertices, $S(G, \Sigma)$ is the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with

1. $a_{i,j} < 0$ if i and j are connected by only even edges,
2. $a_{i,j} > 0$ if i and j are connected by only odd edges,
3. $a_{i,j} \in \mathbb{R}$ if i and j are connected by even and odd edges,
4. $a_{i,j} = 0$ if i and j are not adjacent,
5. $a_{i,i} \in \mathbb{R}$ for all $i \in V(G)$.

Example

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \in S(K_3, \emptyset).$$

The inertia set

Definition

The inertia set $I(G,\Sigma)$ of (G,Σ) is the set of all pairs (p,q) for which there exists an $A \in S(G,\Sigma)$ with exactly p positive and q negative eigenvalues.

Example

$(0,1) \in I(K_3,\emptyset)$.

Theorem

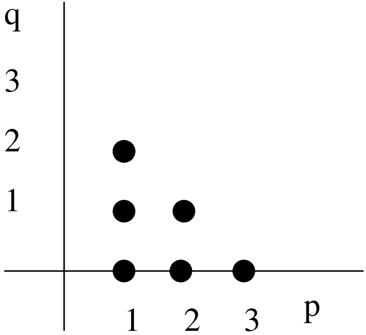
A signature-exchange on (G,Σ) does not change the inertia set.

Northeast Lemma

Lemma

If (G, Σ) has n vertices, $(p, q) \in I(G, \Sigma)$, and $p + q < n$, then $(p + 1, q), (p, q + 1) \in I(G, \Sigma)$.

Shown for graphs by Barrett, Hall, and Loewy.
For $I(K_3, \emptyset)$,



Stable inertia set

The inertia set of a signed graph is difficult to determine. Can we use minors?

Definition

The stable inertia set $I^s(G,\Sigma)$ of (G,Σ) is the set of all pairs (p,q) for which there exists an $A \in S(G,\Sigma)$ that has exactly p positive eigenvalues and q negative eigenvalues and that have the Strong Arnold Hypothesis.

Definition

$T(G)$ denotes the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent.

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Definition

$T(G)$ denotes the set of all symmetric $n \times n$ matrices $A = [a_{i,j}]$ with $a_{i,j} = 0$ if $i \neq j$ and i and j are not adjacent.

Definition

A matrix $A \in S(G, \Sigma)$ satisfies the Strong Arnold Hypothesis (SAH) if for all symmetric matrices K, there exists a matrix $M \in T(G)$ such that $x^T K x = x^T M x$ for all $x \in \ker(A)$.

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \in S(K_{1,4}, \emptyset)$$

Nullity A is 3, but A does not have SAH.

$$(1,1) \in I(K_{1,4}, \emptyset)$$

But

$$(1,1) \notin I^s(K_{1,4}, \emptyset).$$

Theorem

If (G, Σ) has n vertices, $(p, q) \in I^s(G, \Sigma)$, and $p + q < n$, then $(p+1, q), (p, q+1) \in I^s(G, \Sigma)$.

Stable inertia set and subgraphs

- $\mathbb{N}$ includes 0.
- $\mathbb{N}^2_k = \{(p,q) \in \mathbb{N}^2 \mid p+q=k\}$.
- $\mathbb{N}^2_{[k,n]} = \{(p,q) \in \mathbb{N}^2 \mid k \leq p+q \leq n\}$.
- If $A, B \subseteq \mathbb{N}^2$, then
  $A + B = \{(p_1+p_2, q_1+q_2) \mid (p_1,q_1) \in A, (p_2,q_2) \in B\}$.

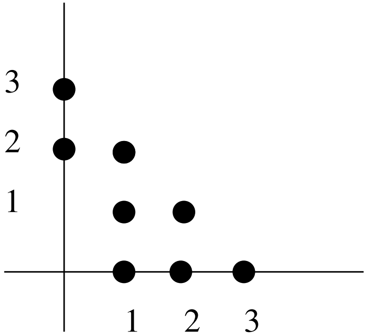
Theorem

Let (G,Σ) be a signed graph with n vertices and $(H, \Sigma \cap E(H))$ a signed subgraph with m vertices. Then $I^s(H, \Sigma \cap E(H)) + \mathbb{N}^2_{n-m} \subseteq I^s(G,\Sigma)$.

Corollary

If (G,Σ) has $n > 0$ vertices, then $\mathbb{N}^2_{[n-1,n]} \subseteq I^s(G,\Sigma) \subseteq I(G,\Sigma)$.

For $(K_3, \emptyset)$,



Stable inertia set and contracting edges

Theorem

Let (G,Σ) be a signed graph and $e \in E(G)$.

1. *If e is odd, then $I^s(G/e, \Sigma \setminus \{e\}) + \{(0,1)\} \subseteq I^s(G,\Sigma)$.*
2. *If e is even, then $I^s(G/e, \Sigma) + \{(1,0)\} \subseteq I^s(G,\Sigma)$.*

Our friend reappears.

What happens if we consider signed graph of the form $(G, \emptyset)$?

Theorem

If G has n vertices, $\mu(G) = \max\{k \geq 0 \mid (n-k-1, 1) \in I^s(G, \emptyset)\}$.

Remember:

Theorem

1. $\mu(G) \leq 3$ *if and only if G is planar.*
2. $\mu(G) \leq 4$ *if and only if G has a flat embedding.*

No characterization known for graphs G with $\mu(G) \leq 5$.

$\nu(G, \Sigma)$

Definition

For a signed graph (G, Σ), $\nu(G, \Sigma)$ is the largest nullity (corank) of any positive semidefinite matrix $A \in S(G, \Sigma)$ satisfying the Strong Arnold Hypothesis.

Observation

$\nu(G) = \max\{\nu(G, \Sigma) \mid \Sigma \subseteq E(G)\}$.

Observation

If G has n vertices, $\nu(G, \Sigma) = \max\{k \geq 0 \mid (n-k, 0) \in I^s(G, \Sigma)\}$.

Problem

Characterize signed graph (G, Σ) with $\nu(G, \Sigma) \leq 1$ and those with $\nu(G, \Sigma) \leq 2$.

Minor-monotonicity of $\nu(G, \Sigma)$

Definition

A signed graph (H, Ω) is a minor of (G, Σ) if (H, Ω) arises from (G, Σ) by a series of deletions of vertices and edges, contractions of even edges, and signature-exchanges.

No contraction of odd edges allowed. First signature-exchange to make the edge even, then contraction.

Theorem

If (H, Ω) is a minor of (G, Σ), then $\nu(H, \Omega) \leq \nu(G, \Sigma)$.

Bipartite signed graphs

Theorem

$\nu(G,\Sigma) \le 1$ if and only if (G,Σ) is bipartite (i.e. has no odd cycles).

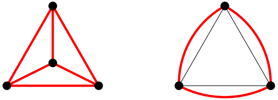
Proof.

- $\nu(C_2,\{e\}) = 2$, so (G,Σ) has no odd cycles.
- For the converse, suppose (G,Σ) has no odd cycle.
- May assume (G,Σ) is connected.
- Apply signature-exchange to make all edges even.
- Perron-Frobenius says that smallest eigenvalue has multiplicity one.

$\square$

Problem

Which signed graphs (G, Σ) have $\nu(G, \Sigma) \leq 2$?



Lemma

$\nu(\text{odd-}K_4) = 3$ *and* $\nu(\text{odd-}C_3^2) = 3$.

Lemma

If $\nu(G, \Sigma) \leq 2$, *then* (G, Σ) *has no odd-*K_4 *and no odd-*C_3^2.

No odd-K_4 and no odd-C_3^2.

Example (of no odd-K_4 and no odd-C_3^2.)

Signed graphs (G, Σ) for which there is a vertex v such that $(G, \Sigma) \setminus v$ is bipartite.

Definition

A signed graph (G, Σ) is almost bipartite if there is a vertex v such that $(G, \Sigma) \setminus v$ is bipartite.

Example

Signed graphs (G, Σ) such that G can be embedded in the plane with at most two odd faces.

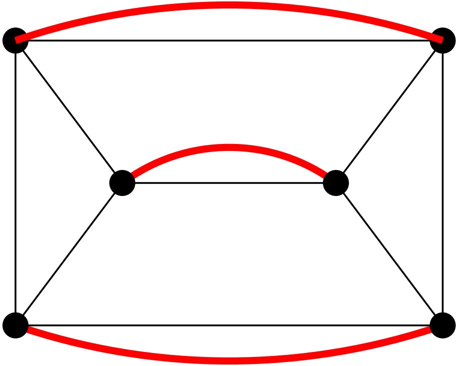


Figure: Prism

Theorem (Gerards)

Let (G, Σ) be a signed graph with no odd-K_4- and no odd-C_3^2-minor. Then at least one of the following holds:

1. (G, Σ) *has a* 0-, 1-, 2-, *or* 3-*split;*
2. (G, Σ) *is almost bipartite;*
3. (G, Σ) *can be embedded in the plane with at most two odd faces;*
4. (G, Σ) *is equivalent to the prism.*

Theorem

If (G_1, Σ_1) *and* (G_2, Σ_2) *form a* 0-, 1-, *or a* 2-*split of* (G, Σ), *then* $\nu(G, \Sigma) = \max\{\nu(G_1, \Sigma_2), \nu(G_2, \Sigma_2)\}$.

Theorem

If (G_1, Σ_1) *is a* 3-*split of* (G, Σ), *then* $\nu(G, \Sigma) = \nu(G_1, \Sigma_1)$.

Hence we need to prove that almost bipartite signed graphs, planar signed graph with two odd faces, and the prism have $\nu(G, \Sigma) \leq 2$.

Almost bipartite

Theorem

If (G, Σ) is almost bipartite, then $\nu(G, \Sigma) \leq 2$.

Proof.

- May assume (G, Σ) is 2-connected.
- Suppose $\nu(G, \Sigma) \geq 3$. Let $A \in S(G, \Sigma)$ be positive semidefinite with nullity ≥ 3.
- Can delete a vertex v such that $(G, \Sigma) \setminus \{v\}$ is connected and bipartite.
- Let w be any other vertex.
- Let $x \in \ker(A)$ be nonzero and with $x_v = x_w = 0$. Perron-Frobenius says restriction of x to $(G, \Sigma) \setminus \{v\}$ must be either all positive or all negative.
- Contradiction.

$\square$

Planar with two odd faces

Theorem

If (G, Σ) is planar with two odd faces, then $\nu(G, \Sigma) \leq 2$.

Proof Idea.

Theorem (Epifanov)

Any planar graph with two terminal can be reduced to a single edge, where in each reduction none of the terminal is involved.

Reductions are: ΔY-exchanges, suppressing degree two vertices, deleting degree one and zero vertices, and removal of all but one edge from each parallel class. Terminals cannot be removed.

- Dualize this theorem. Odd faces $\leftrightarrow$ terminals.
- $\nu(G, \Sigma)$ does not decrease under reductions.
- End up with signed graph $(C_2, \{e\})$. Has $\nu(C_2, \{e\}) = 2$.
- Hence $\nu(G, \Sigma) \leq 2$.

Prism

Theorem

$\nu(prism) = 2$.

Theorem

$\nu(G,\Sigma) \leq 2$ if and only if (G,Σ) has no odd-K_4 and no odd-C_3^2.

Problem (Open)

What about $\nu(G,\Sigma) \leq 3$?

Question

$\nu(G,\Sigma)$ is large if and only if (G,Σ) contains a large super-triangle as a minor.

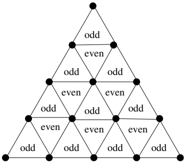


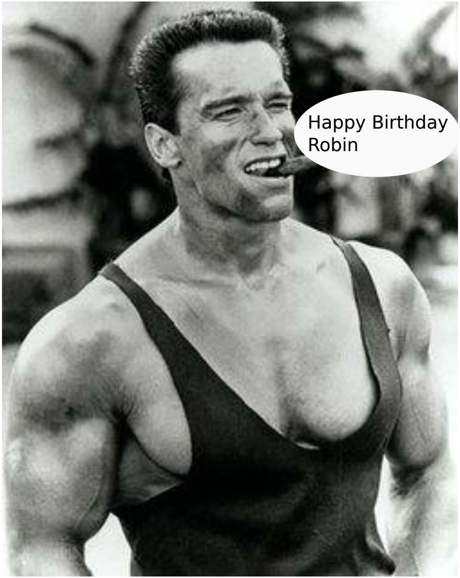
Figure: Supertriangle

Other results

Definition

$\xi(G, \Sigma)$ is the largest nullity of any $A \in S(G, \Sigma)$ satisfying the SAH. (Without the restriction of positive semidefinite.)

We have combinatorial characterization of signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 1$ and of signed graphs (G, Σ) with $\xi(G, \Sigma) \leq 2$.


Happy Birthday
Robin